

Symbolic calculus on the time-frequency half-plane

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Abstract

The study concerns a special symbolic calculus of interest for signal analysis. This calculus associates functions on the time-frequency half-plane $f > 0$ with linear operators defined on the positive-frequency signals. Full attention is given to its construction which is entirely based on the study of the affine group in a simple and direct way. The correspondence rule is detailed and the associated Wigner function is given. Formulas expressing the basic operation (star-bracket) of the Lie algebra of symbols, which is isomorphic to that of the operators, are obtained. In addition, it is shown that the resulting calculus is covariant under a three-parameter group which contains the affine group as subgroup. This observation is the starting point of an investigation leading to a whole class of symbolic calculi which can be considered as modifications of the original one.

1 Introduction

The notion of time-frequency analysis of signals received a theoretical basis when D.Gabor [1] proposed to interpret signals by operations very similar to those of quantum mechanics. In this approach, the real signals are replaced by their positive-frequency parts (the "analytic" signals) and the communication characteristics are introduced as the mean values of hermitian operators defined on the corresponding Hilbert space. Actually, the analogy with quantum mechanics has rapidly suggested to recognize the time and the frequency as two conjugate variables and has led to look for a phase space formulation of the theory. Various solutions have then been proposed which were essentially adaptations of solutions developed precedently in the quantum mechanical context.

From a mathematical point of view, the question of the phase space formulation of a quantum theory appears as a problem of symbolic calculus since each operator of the theory has to be replaced by a function (its symbol) of the phase space variables [2]. The

construction of such a correspondence rule is determined by the class of operators to symbolize and by subsidiary constraints relative to the physical interpretation of the formalism. In signal theory, this last point concerns simply the formal invariance of the operations in changes of reference clocks. The corresponding constraint will be automatically ensured if the correspondence rule is covariant with respect to relevant representations of the affine group. The object of the following developments is to construct a symbolic calculus on the time-frequency half-plane along these lines.

Section 2 gives a study of the affine group and its subgroups, their representations in the Hilbert space \mathcal{H} of signals and their actions in the time-frequency half-plane Γ . Decompositions of \mathcal{H} and Γ into invariant subspaces are carried out in parallel and shown to be related in a natural way. The correspondence rule is derived in Section 3 by exploiting the completeness of the invariant decompositions just performed. Some properties of this rule are given in Section 4 and the associated Wigner function is determined. Section 5 is devoted to the derivation of the star product, which allows to symbolize products of operators and is the core of the symbolic calculus. In Section 6, the star bracket is introduced and shown to reduce to an ordinary Poisson bracket in some special cases. As an application of this remark, a Hamiltonian flow is defined in phase space. Finally, it is shown in Section 7 that the method has possible extensions when working with larger groups containing the affine group.

2 The affine group in signal theory

The signals to be considered are real-valued functions of time $s(t)$ which are interpreted independently of the choice of time origin and units. This means that the description of the signal processing operations must not change if time t is replaced by $at + b$ where b is real and a positive. The corresponding change on the signal can be readily written down. Notice that there is the freedom of rescaling the signal when the units are changed. The transformations on the signal that will be considered are written as:

$$s(t) \longrightarrow a^r s(a^{-1}(t - b)) \quad (2.1)$$

where r is a real exponent.

The set of transformations (2.1) forms the affine group, with parameters (a, b) and composition law expressed by:

$$(a, b)(a', b') = (aa', b + ab') \quad (2.2)$$

The group structure is the mathematical expression of the equivalence of the reference clocks. To stress the fact that only positive dilations are considered, the parameter u such that:

$$a = e^u \quad (2.3)$$

will be used instead of a .

2.1 Representation in the Hilbert space of positive-frequency signals

The Fourier transform of the signal $s(t)$ will be introduced by:

$$\hat{s}(f) \equiv \int_{\mathbb{R}} s(t) e^{-2i\pi ft} dt \quad (2.4)$$

Since $s(t)$ is a real-valued function, it is characterized by its positive frequency part only. In agreement with the usual practice in signal theory, this property will be used to substitute to $s(t)$ the so-called analytic signal whose Fourier transform is given by:

$$S(f) = 2Y(f)\hat{s}(f) \quad (2.5)$$

where $Y(f)$ is Heaviside's step function.

The counterpart of transformation (2.1) for the analytic signal $S(f)$ is easily obtained. Namely, by action of an element (u, b) of the affine group, signal $S(f)$ goes to a new function $[\mathbf{U}(u, b)S](f)$ defined by:

$$\mathbf{U}(u, b)S(f) \equiv e^{(r+1)u} e^{-2i\pi fb} S(e^u f), \quad r \in \mathbb{R} \quad (2.6)$$

The operators $\mathbf{U}(u, b)$ constitute an irreducible representation of the affine group that is unitary for the scalar product defined by

$$(S, S') \equiv \int_{\mathbb{R}^+} S(f) S'^*(f) f^{2r+1} df \quad (2.7)$$

The Hilbert space of functions $S(f)$ on the half-line which are square integrable for the measure $f^{2r+1} df$ will be denoted by \mathcal{H} (the reference to r being implicit). From a physical point of view, it may be necessary to distinguish between signals transforming according to (2.6) with different values of r . However, the corresponding representations are easily shown to be unitarily equivalent. In fact, it may be shown that the affine group has only two inequivalent unitary representations differing by the sign of the imaginary exponential in (2.6) [3].

The infinitesimal operators of representation (2.6) are introduced by the relations:

$$\begin{aligned} \beta S(f) &\equiv -\frac{1}{2i\pi} \frac{d}{du} \mathbf{U}(u, b) S(f) \Big|_{u=b=0} \\ &= -\frac{1}{2i\pi} \left(r + 1 + f \frac{d}{df} \right) S(f) \end{aligned} \quad (2.8)$$

$$\begin{aligned} \mathbf{f} S(f) &\equiv -\frac{1}{2i\pi} \frac{d}{db} \mathbf{U}(u, b) S(f) \Big|_{u=b=0} \\ &= f S(f) \end{aligned} \quad (2.9)$$

The operators β and \mathbf{f} thus obtained verify the commutation relation:

$$[\beta, \mathbf{f}] = -\frac{1}{2i\pi} \mathbf{f} \quad (2.10)$$

The subgroups of the affine group and their representations will now be studied in detail. There are two kinds of subgroups consisting either of pure dilations or of pure translations.

The translation subgroup is invariant by conjugation. On the contrary, since a dilation can be performed from an arbitrary time origin ξ , there is a whole family of conjugate dilation subgroups G_ξ which consist of the following set of elements:

$$G_\xi = \{(e^u, \xi(1 - e^u))\} \quad (2.11)$$

The restriction \mathbf{U}_ξ of representation (2.6) to G_ξ is given by

$$\mathbf{U}_\xi(u)S(f) = e^{(r+1)u} e^{-2i\pi f \xi(1-e^u)} S(e^u f) \quad (2.12)$$

and the corresponding generator is

$$\beta_\xi S(f) \equiv -\frac{1}{2i\pi} \frac{d}{du} \mathbf{U}_\xi(u)S(f) \Big|_{u=0} = (\beta - \xi f)S(f) \quad (2.13)$$

where operators β and f are given by (2.8) and (2.9) respectively.

The operator of representation (2.12) can be written in terms of generator β_ξ as:

$$\mathbf{U}_\xi = e^{-2i\pi u \beta_\xi} \quad (2.14)$$

For a given ξ , the eigenfunctions of the self-adjoint operator β_ξ are defined by the equation

$$\beta_\xi \psi_\beta^\xi(f) = \beta \psi_\beta^\xi(f), \quad \beta \in \mathbb{R} \quad (2.15)$$

whose solution is, up to a multiplicative constant:

$$\psi_\beta^\xi(f) \equiv f^{-2i\pi\beta-r-1} e^{-2i\pi\xi f} \quad (2.16)$$

By construction, the functions $\psi_\beta^\xi(f)$ are also eigenfunctions of the unitary operator (2.14) and verify

$$\mathbf{U}_\xi(u) \psi_\beta^\xi(f) = e^{-2i\pi\beta u} \psi_\beta^\xi(f) \quad (2.17)$$

For a given ξ , functions (2.16) constitute an improper basis of the Hilbert space \mathcal{H} . In fact, one verifies that there is orthogonality of the functions corresponding to different β

$$(\psi_\beta^\xi(f), \psi_{\beta'}^\xi(f)) = \delta(\beta - \beta') \quad (2.18)$$

where the scalar product has the form (2.7). The completeness of this basis is expressed by:

$$\int_{\mathbb{R}} \psi_\beta^\xi(f) \psi_{\beta'}^{\xi*}(f') d\beta = f^{-2r-1} \delta(f - f') \quad (2.19)$$

For each ξ , the set of all the projectors on the functions $\psi_\beta^\xi(f)$ gives a decomposition of the Hilbert space \mathcal{H} . From (2.17) it results that each of these projectors is invariant by the transformations of the subgroup G_ξ .

More generally, the action of the whole affine group on functions $\psi_\beta^\xi(f)$ is given by:

$$\mathbf{U}(u, b) \psi_\beta^\xi(f) = e^{-2i\pi u \beta} \psi_{\beta'}^{\xi'}(f) \quad (2.20)$$

with $\xi' = e^u \xi + b$. This shows that the set of projectors associated with subgroup G_ξ is transformed as a whole into the set of projectors associated with subgroup $G_{e^u \xi + b}$. In other words, this means that the set of all decompositions of \mathcal{H} by the various subgroups G_ξ is invariant by action of the whole group.

2.2 Action of the group in the time-frequency half-plane

The physical phase space for signal analysis is the time-frequency half-plane, $\Gamma = \{(t, f)\}$, t real, $f > 0$. This space can also be obtained as an orbit of the coadjoint representation of the affine group whose action in variables (t, f) is given by:

$$(u, b) : (t, f) \longrightarrow (e^u t + b, e^{-u} f) \quad (2.21)$$

where the pair (u, b) characterizes an element of the group.

This action induces a transformation on functions $\mathcal{A}(t, f)$ defined on Γ . This transformation has the form:

$$(u, b) : \mathcal{A}(t, f) \longrightarrow \mathcal{A}(e^{-u}(t - b), e^u f) \quad (2.22)$$

According to definition (2.11), the restriction of representation (2.21) to subgroup G_ξ reads:

$$(t, f) \longrightarrow (e^u t + \xi(1 - e^u), e^{-u} f) \quad (2.23)$$

and the corresponding transformation of phase space functions is given by:

$$\mathcal{A}(t, f) \longrightarrow \mathcal{A}(e^{-u}(t - \xi) + \xi, e^u f) \quad (2.24)$$

The transformation (2.23) has the following invariant:

$$(t - \xi)f = \tilde{\beta}, \quad \tilde{\beta} \in \mathbb{R} \quad (2.25)$$

where $\tilde{\beta}$ is an arbitrary constant. For each ξ , this relation defines a family of curves in phase space labelled by $\tilde{\beta}$. Actually any point in Γ belongs to one, and only one, of these curves and a partition of phase space has thus been achieved for any value of ξ .

Uniform distributions on curves (2.25) are given by the Dirac distributions:

$$\delta_\beta^\xi \equiv \delta((t - \xi)f - \tilde{\beta}) \quad (2.26)$$

These distributions are invariant by action of the subgroup G_ξ and transform under the full group as follows:

$$(u, b) : \delta_\beta^\xi = \delta((t - \xi)f - \tilde{\beta}) \longrightarrow \delta((e^{-u}(t - b) - \xi)e^u f - \tilde{\beta}) = \delta_\beta^{\xi e^u + b} \quad (2.27)$$

This result can be compared with (2.20). In fact, the full group connects the various phase space partitions associated with subgroups G_ξ .

2.3 Correspondence between invariant structures

The subgroup G_ξ , considered for any given value of ξ , has allowed to perform decompositions of both \mathcal{H} and Γ where the resulting partitions have been labelled by parameters denoted respectively by β and $\tilde{\beta}$. We will now show that a natural connection does exist between these two parameters. To develop this point, it is useful to introduce the notion of localized signals in a consistent way.

A signal $S_{t_0}(f) \in \mathcal{H}$ is said to be localized at time t_0 provided an affine transformation, defined by the element (u, b) of the affine group, sends it into a signal localized at $(e^u t_0 + b)$. This is expressed by the following equation:

$$\mathbf{U}(u, b)S_{t_0}(f) \equiv e^{(r+1)u} e^{-2i\pi f b} S_{t_0}(e^u f) = S_{e^u t_0 + b}(f) \quad (2.28)$$

whose solution is, up to a multiplicative constant:

$$S_{t_0}(f) = f^{-r-1} e^{-2i\pi f t_0} \quad (2.29)$$

Notice the factor f^{-r-1} which is required by dilation invariance. In this way, the whole set of localized states S_{t_0} is stable by action of the affine group.

In the decomposition of \mathcal{H} relative to the subgroup G_ξ , the β -subspace is generated by $\psi_\beta^\xi(f)$ of the form (2.16). Developing the phase of this function in the vicinity of a frequency f_0 leads to the approximate expression:

$$\psi_\beta^\xi(f) \simeq e^{-2i\pi\beta(\ln f_0 - 1)} f^{-r-1} e^{-2i\pi f(\beta/f_0 + \xi)} \quad (2.30)$$

A comparison with (2.29) shows that, apart from a constant phase, the approximated function can be seen as a state localized at:

$$t_0 = \beta/f_0 + \xi \quad (2.31)$$

Thus, in a local study of functions (2.16), it is possible to associate with any frequency f a time t given by $t = \beta/f + \xi$. This relation defines precisely an orbit of subgroup G_ξ in phase space Γ (cf.(2.25)). It leads naturally to identify the parameters of the connected elements by setting:

$$\tilde{\beta} = \beta \quad (2.32)$$

This identification, which is clearly invariant under the group action, will play a central role in the following construction.

3 Geometric correspondence between symbols and operators

The aim now is to set up a correspondence between operators on the Hilbert space \mathcal{H} and functions (their symbols) on the time-frequency half-plane Γ . The approach will be essentially based on the parallel decompositions of \mathcal{H} and Γ introduced in the preceding section. The derived correspondence will be said to be geometric to emphasize the fact that it is essentially founded on a study of the group and of the invariants associated with its subgroups.

3.1 Characterization of operators by their diagonal elements in the various ξ -bases

Consider an operator \mathbf{A} defined by a kernel according to:

$$[\mathbf{A}S](f_1) = \int_{\mathbb{R}^+} A(f_1, f_2) S(f_2) f_2^{2r+1} df_2 \quad (3.1)$$

Its diagonal matrix elements on the G_ξ -invariant basis defined by (2.16) are given by:

$$\begin{aligned} I_{\mathcal{H}}(\xi, \beta) &\equiv (\psi_\beta^\xi, \mathbf{A}\psi_\beta^\xi) \\ &= \int_0^\infty \int_0^\infty \psi_\beta^{\xi*}(f_1) A(f_1, f_2) \psi_\beta^\xi(f_2) (f_1 f_2)^{2r+1} df_1 df_2 \end{aligned} \quad (3.2)$$

Quantities $I_{\mathcal{H}}(\xi, \beta)$ have several properties that will be detailed below. First, for any given value of ξ and β , they are invariant by action of the representation \mathbf{U}_{ξ} on the operator \mathbf{A} . Moreover, the operator \mathbf{A} can be recovered from the values of $I_{\mathcal{H}}(\xi, \beta)$ ($-\infty < \xi < \infty$, $-\infty < \beta < \infty$) and a simple formula exists for the trace of operators. To be able to discriminate between functions $I_{\mathcal{H}}(\xi, \beta)$ associated with distinct operators, we will also use the notation $I_{\mathcal{H}}(\mathbf{A}; \xi, \beta)$ for (3.2).

G_{ξ} -invariance of $I_{\mathcal{H}}(\xi, \beta)$

The matrix elements of the transformed operator $\mathbf{A}' = \mathbf{U}_{\xi}^{-1} \mathbf{A} \mathbf{U}_{\xi}$ are given by :

$$\begin{aligned} (\psi_{\beta}^{\xi}, \mathbf{U}_{\xi}^{-1} \mathbf{A} \mathbf{U}_{\xi} \psi_{\beta}^{\xi}) &= (\mathbf{U}_{\xi} \psi_{\beta}^{\xi}, \mathbf{A} \mathbf{U}_{\xi} \psi_{\beta}^{\xi}) \\ &= (\psi_{\beta}^{\xi}, \mathbf{A} \psi_{\beta}^{\xi}) \end{aligned}$$

the last equality being a consequence of property (2.17).

Affine covariance

The action of the full group on $I_{\mathcal{H}}(\xi, \beta)$ is computed in the same way, using property (2.20). One obtains:

$$I_{\mathcal{H}}(\mathbf{A}'; \xi, \beta) = I_{\mathcal{H}}(\mathbf{A}; e^u \xi + b, \beta) \quad (3.3)$$

where $\mathbf{A}' = \mathbf{U}^{-1}(u, b) \mathbf{A} \mathbf{U}(u, b)$.

This relation expresses the diagonal elements of \mathbf{A}' in the ξ -basis in terms of the diagonal elements of \mathbf{A} in the basis corresponding to the subgroup $G_{\xi'}$, conjugate of G_{ξ} by the transformation (u, b) of the affine group.

Reconstruction of the operator \mathbf{A}

Defining the two-dimensional Fourier transform of $I_{\mathcal{H}}(\xi, \beta)$ by

$$\hat{I}_{\mathcal{H}}(u, v) \equiv \int_{\mathbb{R}^2} e^{-2i\pi(u\xi + v\beta)} I_{\mathcal{H}}(\xi, \beta) d\xi d\beta \quad (3.4)$$

we can compute it from (3.2) and find

$$\hat{I}_{\mathcal{H}}(u, v) = A \left(\frac{ue^{(v/2)}}{2 \sinh(v/2)}, \frac{ue^{(-v/2)}}{2 \sinh(v/2)} \right) \left(\frac{u}{2 \sinh(v/2)} \right)^{2r} \frac{|u|}{4 \sinh^2(v/2)} Y(u/v) \quad (3.5)$$

The inversion of this relation can be done directly by changing the variables. The kernel of operator \mathbf{A} is thus given by:

$$A(f_1, f_2) = \hat{I}_{\mathcal{H}}(f_1 - f_2, \ln(f_1/f_2)) (f_1 f_2)^{-r-1} |f_1 - f_2| \quad (3.6)$$

Trace of a product of two operators

The scalar product of two operators can be defined as

$$\text{Tr}(\mathbf{A} \mathbf{B}^{\dagger}) = \int_{\mathbb{R}^+ \times \mathbb{R}^+} A(f_1, f_2) B^*(f_1, f_2) (f_1 f_2)^{2r+1} df_1 df_2 \quad (3.7)$$

where \mathbf{B}^{\dagger} is the hermitian conjugate of operator \mathbf{B} .

If $\widehat{I}_{\mathcal{H}}(\mathbf{A}; u, v)$ and $\widehat{I}_{\mathcal{H}}(\mathbf{B}; u, v)$ are the functions corresponding respectively to the operators \mathbf{A} and \mathbf{B} , the computation of the trace of \mathbf{AB}^\dagger is performed using (3.6) and the result follows:

$$\text{Tr}(\mathbf{AB}^\dagger) = \int_{\mathbb{R}^2} \widehat{I}_{\mathcal{H}}(\mathbf{A}; u, v) \widehat{I}_{\mathcal{H}}(\mathbf{B}^*; u, v) |u| \, dudv \quad (3.8)$$

3.2 Characterization of symbols by their line integrals on ξ -orbits

The description of functions $\mathcal{A}(t, f)$ defined on phase space Γ can be carried out in a way which parallels that followed for the operators. The starting point is the association of a function $I_\Gamma(\xi, \beta)$ with each phase space function $\mathcal{A}(t, f)$ by the relation:

$$I_\Gamma(\xi, \beta) \equiv \int_{\Gamma} \mathcal{A}(t, f) \delta((t - \xi)f - \beta) \, dt df \quad (3.9)$$

For a given ξ , the function defined by (3.9) is interpreted as the integral of $\mathcal{A}(t, f)$ on the various orbits of the subgroup G_ξ . The study of the properties of function $I_\Gamma(\xi, \beta)$ follows the same steps as in Section 3.1. For practical reasons, the notation $I_\Gamma(\mathcal{A}; \xi, \beta)$ will also be used for expression (3.9).

G_ξ -invariance of $I_\Gamma(\xi, \beta)$

The integral in (3.9) is clearly invariant under the action of G_ξ on $\mathcal{A}(t, f)$ defined by (2.24).

Affine covariance

When $\mathcal{A}(t, f)$ is transformed into $\mathcal{A}'(t, f) \equiv \mathcal{A}(e^u t + b, a^{-1} f)$, the function $I_\Gamma(\mathcal{A}; \xi, \beta)$ becomes:

$$I_\Gamma(\mathcal{A}'; \xi, \beta) = I_\Gamma(\mathcal{A}; e^u \xi + b, \beta) \quad (3.10)$$

This relation connects the integrals of the function \mathcal{A}' on the G_ξ orbits to the integrals of \mathcal{A} on the orbits of the subgroup $G_{\xi'}$, conjugate of G_ξ by the transformation (u, b) of the affine group.

Reconstruction of the function $\mathcal{A}(t, f)$

Relation (3.9) defines in fact the Radon transform of $\mathcal{A}(t, f)$ with respect to arrays of hyperbolas parametrized by ξ and β . To invert this transform, we introduce the two-dimensional Fourier transform of $I_\Gamma(\xi, \beta)$ in a manner analogous to (3.4) and compute

$$\widehat{I}_\Gamma(u, v) = \int_{\mathbb{R}} e^{-2i\pi ut} \mathcal{A}(t, (u/v)) \frac{dt}{|v|} \quad (3.11)$$

The inversion of this formula gives:

$$\mathcal{A}(t, f) = f \int_{\mathbb{R}} e^{2i\pi v f t} \widehat{I}_\Gamma(fv, v) |v| \, dv \quad (3.12)$$

Scalar product of two symbols

Consider two functions $\mathcal{A}(t, f)$, $\mathcal{B}(t, f)$ and their corresponding \widehat{I}_Γ functions. The use of relation (3.12) and Parseval's formula yield:

$$\int_{\Gamma} \mathcal{A}(t, f) \mathcal{B}^*(t, f) \, dt df = \int_{\mathbb{R}^2} \widehat{I}_\Gamma(\mathcal{A}; u, v) \widehat{I}_\Gamma(\mathcal{B}^*; u, v) |u| \, dudv \quad (3.13)$$

3.3 The geometric correspondence rule

So far, operators on the Hilbert space \mathcal{H} and functions on the phase space Γ have been characterized respectively by functions $I_{\mathcal{H}}(\xi, \beta)$ or $I_{\Gamma}(\xi, \beta)$. Moreover, for each value of ξ , both functions $I_{\mathcal{H}}(\xi, \beta)$ and $I_{\Gamma}(\xi, \beta)$ are invariant by action of the subgroup G_{ξ} . Thus it is possible to define the correspondence between operators and symbols by requiring that

$$I_{\mathcal{H}}(\xi, \beta) \equiv I_{\Gamma}(\xi, \beta) \quad (3.14)$$

and to denote the common function by $I(\xi, \beta)$. This relation is stable by the affine group as seen from properties (3.3) and (3.10). Actually, the consistency with the group action implies only the proportionality of the two members of (3.14). The interest of the strict identification which has been adopted is to ensure that the identity operator will be symbolized by the function one.

According to (3.5) and (3.11), the Fourier transform of relation (3.14) leads to:

$$\begin{aligned} \widehat{I}(u, v) &= A \left(\frac{ue^{(v/2)}}{2 \sinh(v/2)}, \frac{ue^{(-v/2)}}{2 \sinh(v/2)} \right) \left(\frac{u}{2 \sinh(v/2)} \right)^{2r} \frac{|u|}{4 \sinh^2(v/2)} Y(u/v) \\ &= \int_{\mathbb{R}} e^{-2i\pi ut} \mathcal{A}(t, (u/v)) \frac{dt}{|v|} \end{aligned}$$

The equality of the two right hand sides gives the relation between the kernel $A(f_1, f_2)$ and the symbol $\mathcal{A}(t, f)$. This is the correspondence rule that can be rewritten either as a formula giving the operator kernel in terms of the symbol (Weyl mapping) or as a formula giving the symbol in terms of the kernel (Wigner correspondence). The result is the following:

Result 3.1 *The operator \mathbf{A} corresponding to the function $\mathcal{A}(t, f)$, defined on the time-frequency half-plane Γ , is given by its kernel (defined in (3.1)) as:*

$$A(f_1, f_2) = (f_1 f_2)^{-r-1} \frac{f_1 - f_2}{\ln(f_1/f_2)} \int_{\mathbb{R}} e^{-2i\pi(f_1 - f_2)t} \mathcal{A}\left(t, \frac{f_1 - f_2}{\ln(f_1/f_2)}\right) dt \quad (3.15)$$

Conversely, the operator \mathbf{A} defined by its kernel $A(f_1, f_2)$, is represented on phase space by a symbol $\mathcal{A}(t, f)$ according to:

$$\mathcal{A}(t, f) = f^{2r+2} \int_{\mathbb{R}} e^{2i\pi v f t} A\left(\frac{f v e^{v/2}}{2 \sinh(v/2)}, \frac{f v e^{-v/2}}{2 \sinh(v/2)}\right) \left(\frac{v}{2 \sinh(v/2)}\right)^{2r+2} dv \quad (3.16)$$

In addition, the following unitarity property holds:

$$\int_{\Gamma} \mathcal{A}(t, f) \mathcal{B}^*(t, f) dt df = \text{Tr}(\mathbf{A} \mathbf{B}^\dagger) \quad (3.17)$$

Among the properties of the correspondence, it can be deduced from (3.16) that the identity operator with kernel:

$$I(f_1, f_2) = (f_1 f_2)^{-r-1/2} \delta(f_1 - f_2) \quad (3.18)$$

has a symbol equal to one, and that operators such that

$$A(f_1, f_2) = A^*(f_2, f_1) \quad (3.19)$$

are in correspondence with real-valued functions $\mathcal{A}(t, f)$.

At this point, it would be desirable to characterize in a precise way classes of operators and functions that are put in correspondence by the above rule. Clearly, because of property (3.17), Hilbert-Schmidt operators can be associated with functions in the space $L^2(\mathbb{R}^+; f^{2r+1}df)$. In fact, it is possible to characterize other classes of operators by their symbols [4]-[5], as is done in the usual Weyl case with bounded operators [6]-[7]. But this is outside the scope of the present paper.

Result (3.1) concerns only functions defined on the positive half-line and their corresponding operators. Some side comments concerning its possible extension are given in the following remark.

Remark 3.1 A direct extension of result (3.1) can be obtained if a function $\mathcal{A}(t, f)$ defined on the whole plane is substituted in (3.15) and if negative values for f_1 and f_2 are allowed. However, the result will be a function $A(f_1, f_2)$ defined only in the first and third quadrant of the (f_1, f_2) plane. The regions where f_1 and f_2 have different signs are definitely forbidden in the above symbolic calculus.

4 Some special cases

4.1 Symbol of the projector on a β -subspace of \mathcal{H}

The β -subspace associated with subgroup G_ξ is characterized by the function $\psi_\beta^\xi(f)$ introduced in (2.16). The projector on this subspace is the linear operator defined by:

$$S(f) \longrightarrow (\psi_\beta^\xi, S) \psi_\beta^\xi(f) \quad (4.1)$$

where the scalar product has the form (2.7). The corresponding kernel is given by (cf. (3.1):

$$\Pi_\beta^\xi(f_1, f_2) = \psi_\beta^\xi(f_1) \psi_\beta^{\xi*}(f_2) \quad (4.2)$$

A direct application of formula (3.16) shows that the symbols associated with (4.2) is the distribution $\delta((t - \xi)f - \beta)$ with support in Γ localized on the β -orbit of subgroup G_ξ .

4.2 Affine Wigner function as a particular symbol

Each signal $S(f)$ defines a real linear functional on the set of hermitian operators by:

$$\langle \mathbf{A} \rangle = (\mathbf{A}S, S) = (S, \mathbf{A}S) \quad (4.3)$$

where the scalar product is given by (2.7).

This functional can also be written as:

$$\langle \mathbf{A} \rangle = \text{Tr}(\Pi_S \mathbf{A}) \quad (4.4)$$

where the trace operation has been defined by (3.7) and where Π_S is the projector on signal $S(f)$ with kernel given by:

$$\Pi_S(f_1, f_2) = S(f_1) S^*(f_2) \quad (4.5)$$

From the reciprocal correspondence (3.15), (3.16) and its properties, it results that each real linear functional on the set of hermitian operators can be expressed as a real linear functional of the real-valued functions on the time-frequency half-plane. The Wigner function for the signal $S(f)$ is defined as the function (or distribution) $\mathcal{P}(t, f)$ on the time-frequency half-plane which allows to rewrite (4.4) in the form:

$$\langle \mathbf{A} \rangle = \int_{\Gamma} \mathcal{P}(t, f) \mathcal{A}(t, f) dt df \quad (4.6)$$

where $\mathcal{A}(t, f)$ is the symbol of operator \mathbf{A} . From this definition, it is clear that the form of the Wigner function is fundamentally dependent of the symbolic calculus which is adopted.

The equality of the right-hand sides of (4.4) and (4.6) gives a relation of type (3.17) (unitarity property) provided the Wigner function $\mathcal{P}(t, f)$ is identified with the geometric symbol of the projector $\mathbf{\Pi}_S$. This makes it possible to obtain the relevant Wigner function by applying the correspondence rule (3.16) to the kernel (4.5). The result is the affine Wigner function [8]:

$$\mathcal{P}(t, f) = f^{2r+2} \int_{\mathbb{R}} e^{2i\pi v f t} S \left(\frac{f v e^{v/2}}{2 \sinh(v/2)} \right) S^* \left(\frac{f v e^{-v/2}}{2 \sinh(v/2)} \right) \left(\frac{v}{2 \sinh(v/2)} \right)^{2r+2} dv \quad (4.7)$$

This function can be introduced in various ways. In the above development, it arises as a consequence of the general correspondence defined by (3.15), (3.16).

Expression (4.7) can also be obtained directly by writing the unitarity relation (3.17) with one symbol equal to $\mathcal{P}(t, f)$ and the other one equal to the localized distribution $\delta(t - t_0) \delta(f - f_0)$. This leads to the relation:

$$\mathcal{P}(t_0, f_0) = \text{Tr} (\mathbf{\Pi}_S \mathbf{\Delta}_{t_0, f_0}) \quad (4.8)$$

where $\mathbf{\Delta}_{t_0, f_0}$ is the operator with symbol $\delta(t - t_0) \delta(f - f_0)$. The kernel of this operator is given by (3.15) and reads:

$$\Delta(f_1, f_2) = (f_1 f_2)^{-r-1} \frac{f_1 - f_2}{\ln f_1 / f_2} e^{-2i\pi(f_1 - f_2)t_0} \delta \left(\frac{f_1 - f_2}{\ln f_1 / f_2} - f_0 \right) \quad (4.9)$$

Function $\mathcal{P}(t, f)$ can then be obtained by (4.8). These results are summarized by:

Result 4.1 *The Wigner function of signal $S(f)$ associated with the geometric correspondence is given by (4.7). It can be obtained either as the symbol of the projector $\mathbf{\Pi}_S$ on signal $S(f)$, or as the mean value (4.8) of the operator $\mathbf{\Delta}_{t_0, f_0}$ whose symbol is $\delta(t - t_0) \delta(f - f_0)$.*

The unitarity property (3.17) gives directly the formulas:

$$\int_{\Gamma} \mathcal{P}(t, f) dt df = \| S \|^2 \quad (4.10)$$

$$\begin{aligned} \int_{\Gamma} \mathcal{P}_1(t, f) \mathcal{P}_2(t, f) dt df &= \text{Tr} (\mathbf{\Pi}_{S_1} \mathbf{\Pi}_{S_2}) \\ &= |(S_1, S_2)|^2 \end{aligned}$$

where \mathcal{P}_1 and \mathcal{P}_2 stand for the Wigner functions corresponding to S_1 and S_2 respectively.

Many other properties could be mentioned which are related to the interpretation of the Wigner function as a time-frequency representation [9], [10].

4.3 Exponentials of generators and Weyl's formulation of the correspondence

We will now show that the symbol of the operator $\mathbf{E}_{u_0 v_0} \equiv e^{-2i\pi(u_0 \beta + v_0 \mathbf{f})}$, where β and \mathbf{f} are the generators defined by (2.8) and (2.9), is given by the function $\mathcal{E}_{u_0 v_0}(t, f) = e^{-2i\pi(u_0 t f + v_0 f)}$.

It follows from the developments of Section 2.1 that the operator $\mathbf{E}_{u_0 v_0}$ satisfies the following eigenvalue equation:

$$e^{-2i\pi(u_0 \beta + v_0 \mathbf{f})} \psi_\beta^{-v_0/u_0}(f) = e^{-2i\pi \beta u_0} \psi_\beta^{-v_0/u_0}(f) \quad (4.11)$$

The kernel of $\mathbf{E}_{u_0 v_0}$ is obtained directly from the spectral decomposition:

$$E_{u_0 v_0}(f_1, f_2) = \int_{\mathbb{R}} e^{-2i\pi \beta u_0} \psi_\beta^{-v_0/u_0}(f_1) \psi_\beta^{*-v_0/u_0}(f_2) d\beta \quad (4.12)$$

and has the explicit expression:

$$E_{u_0 v_0}(f_1, f_2) = e^{(r+1)u_0} \exp\{-2i\pi f_1(v_0/u_0)(e^{u_0} - 1)\} \delta(f_1 e^{u_0} - f_2) f_2^{-2r-1} \quad (4.13)$$

Substituting this result in (3.16), we can write the symbol of $\mathbf{E}_{u_0 v_0}$ as:

$$\begin{aligned} \mathcal{E}_{u_0 v_0}(t, f) &= \int_{\mathbb{R}} e^{2i\pi v f t} e^{(r+1)u_0} \exp\{-2i\pi(f(v_0/u_0)(e^{u_0} - 1)\lambda(v))\} \\ &\quad \times \lambda(-v) e^{(2r+2)(v/2)} \delta(e^{u_0} \lambda(v) - \lambda(-v)) dv \end{aligned} \quad (4.14)$$

where, by definition

$$\lambda(v) = \frac{v e^{v/2}}{2 \sinh(v/2)} \quad (4.15)$$

Since

$$\delta(e^{u_0} \lambda(v) - \lambda(-v)) = \frac{1}{\lambda(u_0)} \delta(v + u_0) \quad (4.16)$$

the result of integration in (4.14) is just:

$$\mathcal{E}_{u_0 v_0}(t, f) = e^{-2i\pi(u_0 f t + v_0 f)} \quad (4.17)$$

Thus, we have established the correspondence

$$\mathcal{E}_{u_0 v_0}(t, f) \equiv e^{-2i\pi(u_0 f t + v_0 f)} \longleftrightarrow \mathbf{E}_{u_0 v_0} = e^{-2i\pi(u_0 \beta + v_0 \mathbf{f})} \quad (4.18)$$

Remark that the variable $\beta = t f$ rather than t appears naturally. It will sometimes be convenient to use it by performing the change of variables according to:

$$\tilde{\mathcal{A}}(\beta, f) = [\mathcal{A}(t, f)]_{t=\beta/f} \quad (4.19)$$

The correspondence (4.18) allows to write the operator \mathbf{A} in terms of its symbol $\mathcal{A}(t, f)$ by the Weyl formula:

$$\mathbf{A} = \int \hat{\mathcal{A}}(u, v) e^{-2i\pi(u \beta + v \mathbf{f})} du dv \quad (4.20)$$

where $\widehat{\mathcal{A}}(u, v)$ is the Fourier transform of $\tilde{\mathcal{A}}(\beta, f)$ given by:

$$\widehat{\mathcal{A}}(u, v) \equiv \int_{\mathbb{R} \times \mathbb{R}^+} e^{2i\pi(u\beta + vf)} \tilde{\mathcal{A}}(\beta, f) d\beta df \quad (4.21)$$

The correspondence between functions and operators defined by (4.20) and (4.21) has a form analogous to Weyl's original rule. In fact, it is what L.Cohen calls Weyl's rule for operators β and f [11]. It can also be obtained as a by-product of Kirillov's analysis [12] as shown by A.Unterberger [4] (see also R.Shenoy and T.Parks [13]).

Remark 4.1 The symbols of the self-adjoint operators β and f are immediately seen to be $\beta = tf$ and f respectively. The time operator, which is especially important in signal analysis, can be defined as the operator with symbol t . A direct computation leads to the result:

$$t = (1/2) (f^{-1}\beta + \beta f^{-1}) \quad (4.22)$$

or:

$$t = -\frac{1}{2i\pi} \left(\frac{r+1/2}{f} + \frac{d}{df} \right) \quad (4.23)$$

Such an operator is not essentially self-adjoint on the Hilbert space $L^2(\mathbb{R}^+; f^{2r+1}df)$ and hence has no conventional spectral decomposition. However, it is known from the deformation quantization theory on phase space [2] that such a difficulty can be easily overcome and does not affect the physical interpretation.

5 Star product on the symbols

Operators on the Hilbert space \mathcal{H} have been mapped to functions on the time-frequency half-plane Γ by the rule (3.16). Since the algebra of operators is non-abelian, the product of operators is mapped into a composition law of functions that cannot be the ordinary product. This situation is familiar from Weyl's calculus where the composition law of functions is known as Moyal's product and is invariant by Heisenberg's group. Analogous products, invariant by different groups, have been constructed by deformation of the usual product and are generally referred to as star products (\star -products) [2]. In the present case, the composition law of symbols corresponding to the product of operators leads directly to a star product that is invariant by the affine group.

In the following, it will be convenient to use another notation for the operator $\exp\{-2i\pi(u\beta + v\mathbf{f})\}$. In fact, according to (4.11) and (4.12), the action of that operator on a signal $S(f)$ can be written as

$$\exp\{-2i\pi(u\beta + v\mathbf{f})\}S(f) = e^{(r+1)u} \exp\{-2i\pi f(v/u)(e^u - 1)\}S(fe^u) \quad (5.1)$$

Comparing this expression with the definition (2.6) of representation \mathbf{U} , we are able to

rewrite (5.1) as ¹:

$$\exp\{-2i\pi(u\beta + v\mathbf{f})\} S(f) = \mathbf{U}\left(u, \frac{v}{u}(e^u - 1)\right) S(f) \quad (5.2)$$

The use of representation $\mathbf{U}(u, b)$ allows to take advantage of the group law and to write the composition of operators as:

$$\mathbf{U}(u, b)\mathbf{U}(u', b') = \mathbf{U}(u + u', b + e^u b') \quad (5.3)$$

It shows directly that the product of two operators of type (5.1) is an operator of the same type. With notation (5.2), the form (4.20) of the correspondence rule (3.15) between functions and operators can be written as:

$$\mathbf{A} = \int_{\mathbb{R}^2} \hat{\mathcal{A}}(u, v) \mathbf{U}\left(u, v \frac{e^u - 1}{u}\right) dudv \quad (5.4)$$

where $\hat{\mathcal{A}}(u, v)$ is defined in (4.21).

Let \mathbf{A} and \mathbf{B} be operators given in terms of their symbols $\hat{\mathcal{A}}(\beta, f)$ and $\hat{\mathcal{B}}(\beta, f)$ by rule (5.4),(4.21). Their product is equal to:

$$\mathbf{AB} = \int_{\mathbb{R}^4} \hat{\mathcal{A}}(u_1, v_1) \hat{\mathcal{B}}(u_2, v_2) \mathbf{U}\left(u_1, v_1 \frac{e^{u_1} - 1}{u_1}\right) \mathbf{U}\left(u_2, v_2 \frac{e^{u_2} - 1}{u_2}\right) du_1 dv_1 du_2 dv_2$$

or, using (5.3) :

$$= \int_{\mathbb{R}^4} \hat{\mathcal{A}}(u_1, v_1) \hat{\mathcal{B}}(u_2, v_2) \mathbf{U}\left(u_1 + u_2, v_1 \frac{e^{u_1} - 1}{u_1} + e^{u_1} v_2 \frac{e^{u_2} - 1}{u_2}\right) du_1 dv_1 du_2 dv_2$$

This product can be rewritten in the form (5.4) of the correspondence rule if variables (u, v) are introduced such that:

$$u = u_1 + u_2, \quad v \frac{e^u - 1}{u} = v_1 \frac{e^{u_1} - 1}{u_1} + e^{u_1} v_2 \frac{e^{u_2} - 1}{u_2} \quad (5.5)$$

The result is:

$$\mathbf{AB} = \int_{\mathbb{R}^2} \hat{\mathcal{C}}(u, v) \mathbf{U}\left(u, v \frac{e^u - 1}{u}\right) dudv \quad (5.6)$$

with $\hat{\mathcal{C}}(u, v)$ defined by:

$$\begin{aligned} \hat{\mathcal{C}}(u, v) &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \hat{\mathcal{A}}(u_1, v_1) \hat{\mathcal{B}}(u_2, v_2) \delta(u - u_1 - u_2) \\ &\quad \times \delta\left(v - \left(v_1 \frac{e^{u_1} - 1}{u_1} + e^{u_1} v_2 \frac{e^{u_2} - 1}{u_2}\right) \frac{u_1 + u_2}{e^{u_1 + u_2} - 1}\right) du_1 dv_1 du_2 dv_2 \end{aligned} \quad (5.7)$$

¹ This result can also be obtained directly by noticing that, in the description of the affine group by 2×2 matrices, the exponentiation of the Lie algebra element defined by (u, v) is equal to:

$$\exp \begin{vmatrix} u & v \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} e^u & (v/u)(e^u - 1) \\ 0 & 1 \end{vmatrix}$$

It follows from (5.6) that $\widehat{\mathcal{C}}(u, v)$ is the Fourier transform of the symbol of the operator **AB**.

To obtain an explicit expression of the star product $\tilde{\mathcal{A}}(\beta, f) \star \tilde{\mathcal{B}}(\beta, f)$, it suffices now to invert the Fourier transform of $\widehat{\mathcal{C}}(u, v)$ and to express the result in terms of $\tilde{\mathcal{A}}(\beta, f)$ and $\tilde{\mathcal{B}}(\beta, f)$. In a first step, we obtain:

$$\tilde{\mathcal{A}}(\beta, f) \star \tilde{\mathcal{B}}(\beta, f) = \int \widehat{\mathcal{A}}(u_1, v_1) \widehat{\mathcal{B}}(u_2, v_2) e^{2i\pi((u_1+u_2)\beta+(v_1+v_2)f)} e^{-2i\pi f h(u_1, v_1, u_2, v_2)} du_1 dv_1 du_2 dv_2 \quad (5.8)$$

where the function h is defined by:

$$h(u_1, v_1, u_2, v_2) \equiv (v_1 + v_2) - \frac{u_1 + u_2}{e^{u_1+u_2} - 1} \left(v_1 \frac{e^{u_1} - 1}{u_1} + e^{u_1} v_2 \frac{e^{u_2} - 1}{u_2} \right) \quad (5.9)$$

or, after some manipulations:

$$h(u_1, v_1, u_2, v_2) = (v_1 u_2 - v_2 u_1) \frac{u_1 e^{u_1} (e^{u_2} - 1) - u_2 (e^{u_1} - 1)}{u_1 u_2 (e^{u_1+u_2} - 1)} \quad (5.10)$$

The exponential of function $h(u_1, v_1, u_2, v_2)$ in (5.8) is what makes the difference with a classical product. It can be replaced by its series expansion so that the integrals in (5.8) can be performed. In the process, the following replacements take place:

$$v_i \longrightarrow \frac{1}{2i\pi} \partial_{f_i}, \quad u_i \longrightarrow \frac{1}{2i\pi} \partial_{\beta_i}, \quad i = 1, 2 \quad (5.11)$$

The development of the star product can thus be written as:

$$\begin{aligned} \tilde{\mathcal{A}}(\beta, f) \star \tilde{\mathcal{B}}(\beta, f) &= \tilde{\mathcal{A}}(\beta, f) \tilde{\mathcal{B}}(\beta, f) \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{f}{4i\pi} (\partial_{f_2} \partial_{\beta_1} - \partial_{f_1} \partial_{\beta_2}) T(\partial_{\beta_1}, \partial_{\beta_2}) \right)^n \tilde{\mathcal{A}}(\beta_1, f_1) \tilde{\mathcal{B}}(\beta_2, f_2) \Big|_{\substack{f_1=f_2=f \\ \beta_1=\beta_2=\beta}} \end{aligned} \quad (5.12)$$

where the contribution of operator T will be discussed below.

If (5.12) is rewritten in terms of t rather than β , using the definition:

$$\beta = tf, \quad \partial_{\beta} = (1/f) \partial_t \quad (5.13)$$

the result is reminiscent of the Weyl product. Actually, the difference lies in the presence of the operator $T(\partial_{\beta_1}, \partial_{\beta_2})$ which comes from the development in powers of u_1 and u_2 of:

$$T(u_1, u_2) = 2 \frac{u_1 e^{u_1} (e^{u_2} - 1) - u_2 (e^{u_1} - 1)}{u_1 u_2 (e^{u_1+u_2} - 1)} \quad (5.14)$$

Operator T introduces derivatives of all orders with respect to β_1 and β_2 . At the lowest orders, one has

$$\begin{aligned} T(\partial_{\beta_1}, \partial_{\beta_2}) &= \\ &1 + \frac{i}{12\pi} (\partial_{\beta_2} - \partial_{\beta_1}) + \frac{1}{12(2\pi)^2} \partial_{\beta_1} \partial_{\beta_2} + \frac{i}{720(2\pi)^3} (\partial_{\beta_2} - \partial_{\beta_1}) (\partial_{\beta_1}^2 + \partial_{\beta_2}^2 + 5\partial_{\beta_1} \partial_{\beta_2}) + \dots \end{aligned} \quad (5.15)$$

As in Weyl's case, the first term in (5.12) is the usual product and the following terms introduce derivatives of the symbols. It can be observed that the star product of two real-valued functions is not real. This is a direct manifestation of the fact that the product of hermitian operators is not hermitian.

Remark 5.1 From Remark 3.1, it follows that in a product of operators, the positive and negative frequency parts are multiplied separately. Hence, the same property will hold for the star product if its expression is extended to symbols defined on the whole plane.

The star product simplifies when one of the symbols depends only on one variable. This will be illustrated by computing its expression when $\tilde{\mathcal{A}}(\beta, f)$ is equal either to β or to a function of f alone and when $\tilde{\mathcal{B}}(\beta, f)$ is the generic distribution:

$$\Delta_{\beta_0, f_0}(\beta, f) = \delta(\beta - \beta_0)\delta(f - f_0) \quad (5.16)$$

with Fourier transform (4.21) given by:

$$\hat{\Delta}_{\beta_0, f_0}(u, v) = e^{-2i\pi(u\beta_0 + vf_0)} \quad (5.17)$$

(i) *Case where $\tilde{\mathcal{A}}(\beta, f) = \beta$*

The Fourier transform of β is :

$$\hat{\mathcal{A}}(u, v) = -\frac{1}{2i\pi}\delta'(u)\delta(v) \quad (5.18)$$

Computation of the star product from (5.8) and (5.10) gives:

$$\begin{aligned} \beta \star \delta(\beta - \beta_0)\delta(f - f_0) = \\ -\frac{1}{2i\pi} \int \delta'(u_1) e^{-2i\pi[u_2(\beta_0 - \beta) + v_2 f_0]} \exp\{2i\pi[u_1\beta + f v_2 \frac{\lambda(u_1 + u_2)}{\lambda(u_2)}]\} du_1 du_2 dv_2 \end{aligned} \quad (5.19)$$

where $\lambda(u)$, defined by (4.15), is such that:

$$\lambda(-u) = e^{-u}\lambda(u) \quad (5.20)$$

Performing the u_1 -integration, we can write (5.19) as:

$$\begin{aligned} \beta \star \delta(\beta - \beta_0)\delta(f - f_0) = \beta\delta(\beta - \beta_0)\delta(f - f_0) \\ + f \int_{\mathbb{R}^2} e^{2i\pi[u_2(\beta - \beta_0) + v_2(f - f_0)]} \left(\frac{d}{du_2}(\ln(\lambda(u_2))) \right) v_2 du_2 dv_2 \end{aligned} \quad (5.21)$$

(ii) *Case where $\tilde{\mathcal{A}}(\beta, f)$ depends only on f .*

The Fourier transform of $\tilde{\mathcal{A}}(\beta, f) = g(f)$, with $g(f)$ an arbitrary function, is equal to:

$$\hat{\mathcal{A}}(u, v) = \delta(u) \int_{\mathbb{R}} g(f) e^{-2i\pi v f} df \quad (5.22)$$

Substitution of the Fourier transforms (5.17) and (5.22) in the expression of the star product (5.8)-(5.10) gives:

$$g(f) \star \delta(\beta - \beta_0)\delta(f - f_0) = \int_{\mathbb{R}^2} g(f\lambda(-u)) e^{2i\pi[u(\beta - \beta_0) + v(f - f_0)]} dudv \quad (5.23)$$

6 Star bracket of symbols and Hamiltonian flow

6.1 Definition of the star bracket

A Lie bracket is now defined on operators by the operation:

$$-2\pi i[\mathbf{A}, \mathbf{B}] = -2\pi i(\mathbf{AB} - \mathbf{BA}) \quad (6.24)$$

This bracket defines a composition law on the subspace of hermitian operators. When transformed by the correspondence rule, using the expression for the star product, this bracket gives rise to an operation on real-valued functions that will be called star bracket and denoted by $\{ , \}_\star$. Thus the star bracket of symbols $\tilde{\mathcal{A}}(\beta, f)$ and $\tilde{\mathcal{B}}(\beta, f)$ is defined by:

$$\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}\}_\star(\beta, f) = -2\pi i(\tilde{\mathcal{A}}(\beta, f) \star \tilde{\mathcal{B}}(\beta, f) - \tilde{\mathcal{B}}(\beta, f) \star \tilde{\mathcal{A}}(\beta, f)) \quad (6.25)$$

According to (5.12), the bracket can be written as;

$$\begin{aligned} \{\tilde{\mathcal{A}}(\beta, f), \tilde{\mathcal{B}}(\beta, f)\}_\star &= \left[f(\partial_{f_1} \partial_{\beta_2} - \partial_{f_2} \partial_{\beta_1})(1 + \frac{1}{12(2\pi)^2} \partial_{\beta_1} \partial_{\beta_2} + \dots) \right. \\ &+ \sum_{n=2}^{\infty} \frac{2i\pi}{n!} \left(\frac{f}{4i\pi} (\partial_{f_1} \partial_{\beta_2} - \partial_{f_2} \partial_{\beta_1}) \right)^n \left[T^n(\partial_{\beta_2}, \partial_{\beta_1}) - (-1)^n T^n(\partial_{\beta_1}, \partial_{\beta_2}) \right] \\ &\left. \times \tilde{\mathcal{A}}(\beta_1, f_1) \tilde{\mathcal{B}}(\beta_2, f_2) \Big|_{\substack{f_1=f_2=f \\ \beta_1=\beta_2=\beta}} \right] \end{aligned} \quad (6.26)$$

The first term in f is a Poisson bracket given by:

$$\{\tilde{\mathcal{A}}(\beta, f), \tilde{\mathcal{B}}(\beta, f)\}_P = f \left(\frac{\partial \tilde{\mathcal{A}}(\beta, f)}{\partial \beta} \frac{\partial \tilde{\mathcal{B}}(\beta, f)}{\partial f} - \frac{\partial \tilde{\mathcal{A}}(\beta, f)}{\partial f} \frac{\partial \tilde{\mathcal{B}}(\beta, f)}{\partial \beta} \right) \quad (6.27)$$

In fact, if the variable β is replaced by its definition (5.13), the expression (6.27) becomes identical with the usual Poisson bracket on Γ . This is the same bracket as obtained by Kirillov's theory [12] when considering phase space as an orbit of the coadjoint representation.

The form (6.26) of the star bracket is reminiscent of that obtained with the Weyl calculus. However, due to the fact that operator $T(\partial_{\beta_1}, \partial_{\beta_2})$ has no symmetry, derivatives of arbitrary order will in general appear in the expression of the bracket.

Some symbols are such that their star bracket with any other function has the form of a Poisson bracket. In deformation quantization, such symbols are called *preferred observables* [2], [14], [15]. A typical example is the harmonic oscillator Hamiltonian on \mathbb{R}^2 with Weyl quantization.

In the present case, the preferred observable are represented by the symbols β , f and $\ln f$. Explicitly, their star brackets with any symbol $\mathcal{X}(\beta, f)$ are given by:

$$\{\beta, \mathcal{X}(\beta, f)\}_\star = f \frac{\partial}{\partial f} \mathcal{X}(\beta, f) \quad (6.28)$$

$$\{f, \mathcal{X}(\beta, f)\}_\star = -f \frac{\partial}{\partial \beta} \mathcal{X}(\beta, f) \quad (6.29)$$

$$\{\ln f, \mathcal{X}(\beta, f)\}_\star = -\frac{\partial}{\partial \beta} \mathcal{X}(\beta, f) \quad (6.30)$$

The derivation of these formulas can be based on expressions (5.21), (5.23) and on their counterpart with reverse order in the star products. Final results follow from the relations:

$$\lambda(u) - \lambda(-u) = u, \quad \lambda(-u) = e^{-u} \lambda(u) \quad (6.31)$$

which are verified by the function (4.15)

6.2 Extended covariance and Hamiltonian flows

Consider a one-parameter group of transformations on signals $S(f)$ defined by an operator \mathbf{O} such that:

$$S(f; \alpha) = e^{-2i\pi\alpha\mathbf{O}} S(f; 0) \quad (6.32)$$

with $S(f; 0) = S_0(f)$ given. Signal $S(f; \alpha)$ verifies the equation:

$$\frac{\partial S(f; \alpha)}{\partial \alpha} = -2i\pi\alpha \mathbf{O} S(f; \alpha) \quad (6.33)$$

and the projector $\mathbf{\Pi}$ on $S(f; \alpha)$ verifies:

$$\frac{\partial \mathbf{\Pi}}{\partial \alpha} = -2i\pi[\mathbf{O}, \mathbf{\Pi}] \quad (6.34)$$

Now, going to phase space, we know that the geometric symbol of $\mathbf{\Pi}$ is the affine Wigner function (4.7) for signal $S(f; \alpha)$, and that the symbol of the bracket $-2i\pi[\ , \]$ is the star bracket $\{ \ , \ }_\star$. Thus equation (6.34) becomes the following equation in Γ :

$$\frac{\partial \mathcal{P}(t, f; \alpha)}{\partial \alpha} = \{\mathcal{O}(t, f), \mathcal{P}(t, f; \alpha)\}_\star \quad (6.35)$$

where $\mathcal{O}(t, f)$ is the symbol of operator \mathbf{O} .

Consider now the case where the symbol $\mathcal{O}(t, f)$ has the particular form:

$$\mathcal{O}_H(t, f) = \mu t f + \nu f + \sigma \ln f \quad (6.36)$$

where μ , ν and σ are real constants. From the values of the star brackets (6.28), (6.29) and (6.30), it follows that the bracket in (6.35) reduces to Poisson's bracket. As a result, equation (6.35) takes the form of a Liouville equation with Hamiltonian equal to $\mathcal{O}_H(t, f)$:

$$\begin{aligned} \frac{\partial \mathcal{P}(t, f; \alpha)}{\partial \alpha} &= \{\mu t f + \nu f + \sigma \ln f, \mathcal{P}(t, f; \alpha)\}_P \\ &= \mu f \frac{\partial \mathcal{P}(t, f; \alpha)}{\partial f} - (\mu t + \nu + (\sigma/f)) \frac{\partial \mathcal{P}(t, f; \alpha)}{\partial t} \end{aligned} \quad (6.37)$$

This means that the α -evolution of the Wigner function is identical to that of an incompressible fluid in the time-frequency half-plane. Integration of equation (6.37) with the initial condition

$$\mathcal{P}(t, f; 0) = \mathcal{P}_0(t, f) \quad (6.38)$$

gives:

$$\mathcal{P}(t, f; \alpha) = \mathcal{P}_0 \left((-\nu\mu^{-1} + (t + \nu\mu^{-1}) e^{-\mu\alpha} - \sigma f^{-1} \alpha), f e^{\mu\alpha} \right) \quad (6.39)$$

The evolution of the Wigner function in terms of parameter α has a counterpart in Hilbert space that can be obtained using the symbolic calculus. The operator \mathbf{O}_H for this evolution is found directly from (4.20)-(4.21) and has the expression:

$$\mathbf{O}_H = \mu\beta + \nu\mathbf{f} + \sigma \ln \mathbf{f} \quad (6.40)$$

where the operators β and \mathbf{f} have been introduced in (2.8) and (2.9). The expression (6.32) written for the operator \mathbf{O}_H then gives the α -evolution in the Hilbert space which corresponds to the phase space evolution (6.39). In fact, this leads to a special form of a general transformation which can be written as:

$$(u, b, c) : S(f) \longrightarrow S'(f) = e^{(r+1)u} e^{-2i\pi bf} f^{-2i\pi c} S(e^u f), \quad u, b, c \in \mathbb{R} \quad (6.41)$$

This transformation is a projective unitary representation of a three-parameter group G_0 which has been studied in [16]. It has been proved that transformations (6.41) correspond to symplectic transformations in the phase space. This extended covariance of the calculus is at the basis of a generalization of the construction process that will now be considered.

7 Other symbolizations with geometrical features.

The method presented above to construct a symbolic calculus on the affine group can be embedded in a more general approach based on the systematic consideration of groups containing the affine group and having the time-frequency half-plane as coadjoint orbit. Such groups belong to the class of simply connected Lie groups having a coadjoint orbit of dimension smaller or equal to two [18]. In this class, the groups of interest are $SL(2, \mathbb{R})$ and the three-parameter solvable groups with Lie algebras defined by generators X_a , X_b and X_c satisfying [17]:

$$[X_a, X_b] = X_b, \quad [X_a, X_c] = k X_c, \quad k \in \mathbb{R} \quad (7.42)$$

$$[X_a, X_b] = X_b, \quad [X_a, X_c] = X_c + X_b \quad (7.43)$$

Examples of symbolic calculi invariant by $SL(2, \mathbb{R})$ have been thoroughly studied in [19]-[20]. Here, we will treat the cases of groups G_k with Lie algebras defined by relations (7.42) and (7.43) ². They provide a natural generalization of the group G_0 considered so far in this paper and will be associated with new forms of symbolic calculus. All groups G_k have been introduced in [16] as a tool to find explicit forms of time-frequency pseudo-distributions. We refer to that paper for the details concerning the precise construction of their unitary representations (projective or not) according to the values of k . Here we need only recall the results corresponding to values of k different from 0 and 1.

The groups G_k , $k \neq 0, 1$ consist of elements (u, b, c) with the composition law:

$$gg' = (u + u', b + e^u b', c + e^{ku} c') \quad (7.44)$$

Their unitary representations in the Hilbert space \mathcal{H} defined in Section 2.1 are given by:

$$\mathbf{U}_k(u, b, c)S(f) = e^{(r+1)u} e^{-2i\pi(bf + cf^k)} S(e^u f) \quad (7.45)$$

²The notation G_1 will designate the group with Lie algebra defined in (7.43).

The action of the groups in the time-frequency half-plane Γ is:

$$(t, f) \longrightarrow (e^u t + b + kce^{-(k-1)u} f^{k-1}, e^{-u} f) \quad (7.46)$$

It is clear from (7.45) and (7.46) that the cases $k = 0$ and $k = 1$ are degenerate and deserve a special treatment (see [16]).

Following the same pattern as in section 2, we determine the subgroups of G_k that are conjugate to the dilation subgroup. They are labelled by two real parameters ξ, η and consist of:

$$G_{\xi\eta} = \{(e^u, \xi(1 - e^u), \eta(1 - e^{ku}))\} \quad (7.47)$$

The restriction of representations (7.45) to subgroups $G_{\xi\eta}$ is:

$$\mathbf{U}_{\xi\eta} S(f) = e^{(r+1)u} e^{-2i\pi(\xi(1-e^u)f + \eta(1-e^{ku})f^k)} S(f e^u) \quad (7.48)$$

The eigenfunctions of $\mathbf{U}_{\xi\eta}$ are:

$$\psi_{\beta}^{\xi\eta}(f) = f^{-2i\pi\beta-r-1} e^{-2i\pi(\xi f + \eta f^k)} \quad (7.49)$$

They transform under the full group G_k as:

$$\mathbf{U}(u, b, c) \psi_{\beta}^{\xi\eta}(f) = e^{-2i\pi\beta u} \psi_{\beta}^{\xi e^u + b, \eta e^{ku} + c}(f) \quad (7.50)$$

In phase space Γ , the curves invariant by action (7.46) restricted to $G_{\xi\eta}$ are given by:

$$(t - \xi)f - k\eta f^k = \tilde{\beta} \quad (7.51)$$

A local study of $\psi_{\beta}^{\xi\eta}(f)$ analogous to that performed in Section 2.3 leads to the identification:

$$\tilde{\beta} = \beta \quad (7.52)$$

As in Section 2, we have achieved a correspondence between invariant structures in \mathcal{H} and Γ . But we now have one extra parameter labelling the vectors $\psi_{\beta}^{\xi\eta}(f)$ and the curves (7.51). To proceed with the construction, we must fix one of these parameters in such a way that the affine covariance of the procedure is preserved. Because of the transformation law (7.50), the only possible choices are either $\eta = 0$ or $\beta = \beta_0$ fixed. The case $\eta = 0$ yields the calculus just developed in the preceding sections. The only possibility to obtain a new calculus is to fix the value of β and to let η free.

The same geometric construction as in Section 3 will be carried out. Consider an operator \mathbf{A} on \mathcal{H} and form its diagonal matrix elements on $\psi_{\beta_0}^{\xi\eta}(f)$:

$$I_{\mathcal{H}}^{\beta_0}(\xi, \eta) = (\psi_{\beta_0}^{\xi\eta}, \mathbf{A} \psi_{\beta_0}^{\xi\eta}) \quad (7.53)$$

Consider a function $\mathcal{A}(t, f)$ and form its integral with respect to curves (7.51):

$$I_{\Gamma}^{\beta_0}(\xi, \eta) = \int_{\Gamma} \mathcal{A}(t, f) \delta((t - \xi)f - k\eta f^k - \beta_0) dt df \quad (7.54)$$

The study of the properties of functions $I_{\mathcal{H}}^{\beta_0}(\xi, \eta)$ and $I_{\Gamma}^{\beta_0}(\xi, \eta)$ follow the same steps as in Section 3.

$G_{\xi\eta}$ -invariance of $I_{\mathcal{H}}^{\beta_0}(\xi, \eta)$ and $I_{\Gamma}^{\beta_0}(\xi, \eta)$.

It can be readily verified, using (7.47), (7.50) and the definitions of $I_{\mathcal{H}}^{\beta_0}(\xi, \eta)$ and $I_{\Gamma}^{\beta_0}(\xi, \eta)$.

G_k -covariance

If $\mathbf{A}' = \mathbf{U}_k^{-1}(u, b, c)\mathbf{A}\mathbf{U}_k(u, b, c)$ is the operator transformed from \mathbf{A} by the group representation (7.45), it follows from (7.50) that:

$$I_{\mathcal{H}}^{\beta_0}(\mathbf{A}'; \xi, \eta) = I_{\mathcal{H}}^{\beta_0}(\mathbf{A}; e^u \xi + b, e^{ku} \eta + c) \quad (7.55)$$

On the other hand, if $\mathcal{A}'(t, f) = \mathcal{A}(e^u t + b + kce^{-(k-1)u} f^{k-1}, a^{-1} f)$, a direct computation leads to:

$$I_{\Gamma}^{\beta_0}(\mathcal{A}'; \xi, \eta) = I_{\Gamma}^{\beta_0}(\mathcal{A}; e^u \xi + b, e^{ku} \eta + c) \quad (7.56)$$

Possibility of reconstruction of the operator \mathbf{A} and of the function $\mathcal{A}(t, f)$.

In spite of the fact that the functions $\psi_{\beta_0}^{\xi\eta}(f)$ do not form a basis, it can be shown that the operator \mathbf{A} is completely characterized by its diagonal elements (7.53).

The reconstruction of the function $\mathcal{A}(t, f)$ could be performed by inverting the Radon transform (7.54).

These properties allow to base the correspondence rule on the identification:

$$I_{\mathcal{H}}^{\beta_0}(\xi, \eta) = I_{\Gamma}^{\beta_0}(\xi, \eta) \quad (7.57)$$

which can be seen as an extension of (3.14). The resulting correspondence rule has the form:

$$\mathcal{A}(t, f) = \quad (7.58)$$

$$f^{2r+2} \int_{\mathbb{R}} e^{2i\pi u \beta_0} e^{2i\pi(tf - \beta_0)(\lambda_k(u) - \lambda_k(-u))} A(f\lambda_k(u), f\lambda_k(-u)) (\lambda_k(u)\lambda_k(-u))^{r+1} du$$

where the functions $\lambda_k(u)$ are defined by

$$\lambda_k(u) = e^{u/2} \left(k \frac{\sinh(u/2)}{\sinh(ku/2)} \right)^{1/(k-1)} \quad (7.59)$$

Conversely the kernel of the operator \mathbf{A} can be expressed in terms of the symbol by:

$$\begin{aligned} A(f_1, f_2) &= \int e^{-2i\pi(f_1 - f_2)(t - (\beta_0/f))} \delta(f_1 - f_2 - k^{-1} f^{1-k} (f_1^k - f_2^k)) \\ &\times \left(\frac{f_1}{f_2} \right)^{-2i\pi\beta_0} (f_1 f_2)^{-r} f^{-k} |f_2^{k-1} - f_1^{k-1}| \mathcal{A}(t, f) dt df \end{aligned} \quad (7.60)$$

The operator \mathbf{A} corresponding to this kernel can be written in Weyl's form as:

$$\mathbf{A} = \int_{\mathbb{R}^2} \hat{\mathcal{A}}(k, \beta_0; u, v) e^{-2i\pi(u\beta + v\mathbf{f})} dudv \quad (7.61)$$

where

$$\begin{aligned} \hat{\mathcal{A}}(k, \beta_0; u, v) &= e^u \left| \frac{1 - e^{(k-1)u}}{(1-k)u} \right| (\lambda_k(-u))^{k+1} e^{2i\pi\beta_0(u-\lambda_k(u)+\lambda_k(-u))} \\ &\times \int_{\Gamma} e^{2i\pi t f(\lambda_k(u)-\lambda_k(-u))} e^{2i\pi f(v/u)\lambda_k(-u)(e^u-1)} \mathcal{A}(t, f) f dt df \end{aligned} \quad (7.62)$$

It may be noticed that the limit for $k = 0$ of function $\lambda_k(u)$ is identical to $\lambda(u)$ defined in (4.15). In that case, the relation (7.62) becomes a Fourier transform with respect to $\beta = tf$ and f and the rule (4.20)-(4.21) is recovered. In the general case, the Fourier transform (4.21) is replaced by the k -dependent transform (7.62) for which Parseval's formula does not hold. As a consequence, the unitarity property (3.17) cannot be obtained.

However, it is possible to recover an expression for the scalar product of operators in terms of symbols. This is accomplished by associating two symbols with any operator \mathbf{A} : the symbol $\mathcal{A}(t, f)$ defined by (7.58) and a dual symbol $\mathcal{A}_d(t, f)$ defined by:

$$\begin{aligned} \mathcal{A}_d(t, f) &= \\ f^{2r+2} \int_{\mathbb{R}} e^{2i\pi u \beta_0} e^{2i\pi(tf-\beta_0)(\lambda_k(u)-\lambda_k(-u))} A(f\lambda_k(u), f\lambda_k(-u)) \mu_d(u) du \end{aligned} \quad (7.63)$$

where

$$\mu_d(u) = (\lambda_k(u)\lambda_k(-u))^{r+1} \left| \frac{d}{du}(\lambda_k(u) - \lambda_k(-u)) \right| \quad (7.64)$$

It can be observed that it is only in the case $k = 0$ that:

$$\frac{d}{du}(\lambda_k(u) - \lambda_k(-u)) = 1 \quad (7.65)$$

leading to

$$\mathcal{A}(t, f) = \mathcal{A}_d(t, f) \quad (7.66)$$

For an arbitrary value of k , the unitarity formula is replaced by:

$$\text{Tr}(\mathbf{A}\mathbf{B}^\dagger) = \int_{\Gamma} \mathcal{A}(t, f) \mathcal{B}_d^*(t, f) dt df \quad (7.67)$$

where $\mathcal{A}(t, f)$ is the symbol of \mathbf{A} and $\mathcal{B}_d(t, f)$ is the dual symbol of \mathbf{B} .

These different correspondence rules may be of interest for the study of special operators. The case $k = -1$ where

$$\lambda_{-1}(u) = e^{u/2} \quad (7.68)$$

stands out. The symbolic calculus corresponding to $k = -1$ and $\beta_0 = 0$ has been introduced directly by A.Unterberger [21] who applied it in a mathematical context.

A Wigner function $\mathcal{P}_k^{\beta_0}(t, f)$ can be associated with each k -calculus. Because of the form of relation (7.67), the Wigner function of signal $S(f)$ is defined as the dual symbol of the projector on the signal. We thus associate a time-frequency distribution with each value of (k, β_0) . When $\beta_0 = 0$, these distributions coincide with the generalized passive distributions introduced in [16]. The case $k = 1$, which has not been treated explicitly here, can be deduced by continuity.

8 Conclusion

A correspondence rule between functions on the time-frequency half-plane and operators on the Hilbert space of positive-frequency signals has been obtained. The rule, which is said geometric, is entirely based on the study of the affine group and more specially on the representations of its subgroups in the two domains. It is obtained, relatively to each subgroup, by identification of the decomposition in invariant subspaces with the corresponding tomographic decomposition in the time-frequency half-plane. The construction is very natural and the result can be considered as an affine version of the Weyl rule introduced in quantum mechanics.

In the correspondence, a Lie algebra of hermitian operators is transformed into a Lie algebra of symbols, also called star algebra. The basic operations of this algebra are the star product and the star bracket of symbols. Formulas for these operations have been given. A special attention has been paid to the case where the star bracket reduces to a Poisson bracket.

The obtained symbolic calculus has been integrated into a family of calculi covariant by three-parameter groups G_k , $k \in \mathbb{R}$, containing the affine group. In this family, the geometric rule corresponds to the special value $k = 0$ and is the only one to ensure the unitarity property.

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